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COMPARISON THEOREMS FOR THE VOLUME OF A COMPLEX SUBMANIFOLD OF A KAEHLER MANIFOLD

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ABSTRACT

Let M be a Kaehler manifold of real dimension $2n$ with holomorphic sectional curvature $K_H \geq 4\lambda$ and antiholomorphic Ricci curvature $\rho_A \geq (2n-2)\lambda$, and P is a complex hypersurface. We give a bound for the quotient (volume of P)/(volume of M) and prove that this bound is attained if and only if $P = \mathbf{C}P^{n-1}(\lambda)$ and $M = \mathbf{C}P^n(\lambda)$. Moreover, we give some results on the volume of tubes about P in M.

§1. Introduction

In [H–K], Heintze and Karcher obtained, for the volume of a submanifold of a Riemannian manifold, an inequality which has, among others, the following interesting consequence: "Let $\mathcal{R}_{n,\lambda,\Lambda}$ be the family of pairs (P,M) such that M is an n-dimensional connected compact Riemannian manifold with Ricci curvature bounded from below by $(n-1)\lambda$ $(\lambda > 0)$ and P is a connected compact hypersurface of M with Λ as an upper bound of the norm of the mean curvature of P. Then the function 'relative volume' $\mathcal{V}: \mathcal{R}_{n,\lambda,\Lambda} \to \mathbf{R}$ defined by $\mathcal{V}(P,M) = (\mathrm{vol}(P))/(\mathrm{vol}(M))$ has a minimum on the pair $(S^{n-1}_{\lambda+\Lambda^2}, S^n_\lambda)$, and this is the only pair on which the minimum is attained". Here S^n_μ denotes the q-sphere of constant sectional curvature μ. A similar result is obtained for P of arbitrary codimension if the hypothesis on the Ricci curvature is changed by one on the sectional curvature.

In [G–M], V. Miquel and the author have considered the smaller family $\mathcal{K}_{n,\lambda,h,k}$

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of the pairs (P, M) such that M is a connected compact Kaehler manifold of real dimension $2n$ with antiholomorphic Ricci curvature and holomorphic sectional curvature bounded from below by $(2n-2)\lambda$ and 4λ respectively, and P is a real hypersurface of M with h and k as upper bounds of the modulus of the JN-mean curvature H_{JN} and the JN-normal curvature k_{JN} respectively and with $k_{JN} H_{JN} \geq 0$. We have proved that the relative volume has a minimum on the pair $(S_r^{\mathbf{C}}, \mathbf{C}P^n(\lambda))$ and this is the only pair on which this minimum is attained. Here $S_r^{\mathbf{C}}$ is the geodesic sphere of radius r of the complex projective space $\mathbf{C}P^n(\lambda)$ of real dimension $2n$ and constant holomorphic sectional curvature 4λ; r is given as a function of λ, h and k.

The aim of this note is to obtain bounds for the function $\mathcal{V}$ on other sets of pairs (submanifold, Riemannian manifold) and the characterization of some pairs by the fact that they realize one of these bounds. In particular, the main result we prove here is: "Let $\mathfrak{M}_{n,\lambda}$ be the family of pairs (P, M) where M is a Kaehler manifold of real dimension $2n$ with holomorphic sectional curvature $K_H \geq 4\lambda$ and antiholomorphic Ricci curvature $\rho_A \geq (2n-2)\lambda$, and P is a complex hypersurface of M. Then $\mathcal{V}$ has a minimum on the pair $(\mathbf{C}P^{n-1}(\lambda), \mathbf{C}P^n(\lambda))$ and this is the only pair on which the minimum is attained".

The plan of the paper is the following. In §2 we define the Jacobi operator for submanifolds of arbitrary codimension and give an expression of the metric tensor of the manifold in terms of this operator. This will be a fundamental tool for the characterization of the pairs realizing a bound. In §3 we prove the main theorem of the paper. In §4 we give a theorem for tubes around complex submanifolds in Kaehler manifolds which is in the spirit of the theorems of Gromov ([Gv, lemma 5.3 bis]) and Nayatani ([Na, th. 1]), and which generalizes a result of A. Gray ([Gr 2, th. 6.1]) in the same vein that Gromov's result generalizes one of Bishop ([B–C, p. 253]). We get some consequences of this theorem by using, in an essential manner, some technical results given in §2 and §3.

Now, we recall some definitions and give some notation.

Let $(M, \langle\ ,\ \rangle, J)$ be a Kaehler manifold of real dimension $2n$. A holomorphic plane is that generated by two vectors of the form X, JX. An antiholomorphic plane is that generated by two vectors X, Y such that Y is orthogonal to both X and JX.

The holomorphic (antiholomorphic) sectional curvature $K_H(K_A)$ of $(M, \langle\ ,\ \rangle, J)$ is the restriction of the sectional curvature of M to the holomorphic (antiholomorphic) planes. We shall denote by $K_H(X)$ the holomorphic sectional curvature of the plane generated by X and JX.

We shall adopt the following definition for the curvature and the Riemann–Christoffel tensor:

$$R(X,Y)Z = -[\nabla_X,\nabla_Y]Z + \nabla_{[X,Y]}Z \quad \text{and} \quad R_{XYZW} = \langle R(X,Y)Z,W\rangle.$$

Given a point $p \in M$, a vector $X \in T_pM$ and a holomorphic subspace Π of real dimension $2q$ and orthogonal to X, the *antiholomorphic q-sectional curvature* $K(X,\Pi)$ of X at Π, is defined by

$$K(X,\Pi) = \sum_{i=1}^{2q} R_{Xe_iXe_i},$$

where $\{e_1, Je_1 = e_2, \ldots, e_{2q-1}, Je_{2q-1} = e_{2q}\}$ is a J-orthonormal basis of Π. This concept is just the restriction of the $2q$-mean curvature (defined in [B–C, page 253]) to the holomorphic planes. Then $K(X,\Pi)$ depends only on the $2q$-plane Π and on X. We note that, as a consequence of the definition, Π must be also orthogonal to JX. When $q = n-1$, the plane Π is uniquely determined, then $K(X,\Pi)$ depends only on X, and it will be denoted by $\rho_A(X)$ and called the *antiholomorphic Ricci curvature* of X.

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§2. Jacobi operators

A very useful tool to characterize the pair on which the minimum of the relative volume is attained is the Jacobi operator. It is defined in [Ch] for real hypersurfaces and for points (see also [V–W]). Here we extend this concept to submanifolds of arbitrary real codimension.

Let M be a Riemannian manifold of dimension n, P a submanifold of M of dimension $q < n$. We shall denote by ($\mathfrak{s}\mathfrak{N}P$) $\mathfrak{N}P$ the (unit) normal bundle of P in M, and by $\mathfrak{N}_pP$ ($\mathfrak{s}\mathfrak{N}_pP$) the fiber of $\mathfrak{N}P$ ($\mathfrak{s}\mathfrak{N}P$) at $p \in P$. For every $N \in \mathfrak{s}\mathfrak{N}P$, L_N will denote the Weingarten map of P associated to N.

2.1. Definition. Let $p \in P \subset M$ and $N \in \mathfrak{s}\mathfrak{N}_pP$. Let $\gamma_N(t)$ be the geodesic of M satisfying $\gamma_N(0) = p$, $\gamma_N'(0) = N$. For every $e \in T_pM$, let e^T be the component of e in T_pP and $e^\perp$ the component of e in $\mathfrak{N}_pP$. Let us denote by $\{N\}^\perp$ the orthogonal complement of the vector space generated by N in T_pM. Then the Jacobi operator $\mathfrak{A}(t,p,N)$ is the map

$$\mathfrak{A}(t,p,N) : \{N\}^\perp \to \{N\}^\perp$$

defined by

$$\mathcal{A}(t,p,N)e = \tau_t^{-1} Y(t),$$

where $Y(t)$ is the transverse Jacobi field along γ such that $Y(0) = e^T$ and $\nabla_N Y + L_N e^T = e^\perp$, and τ_t is the parallel transport along $\gamma_N(t)$.

Let us denote by $\mathcal{R}(t)$ the operator $\mathcal{R}(t): \{N\}^\perp \to \{N\}^\perp$ defined by $\mathcal{R}(t) = \tau_t^{-1} \circ R(t) \circ \tau_t$, where $R(t)$ is given by $R(t)U = R(\gamma_N'(t), U)\gamma_N'(t)$ for every $U \in T_{\gamma_N(t)}M$.

By a direct computation from Definition 2.1 one gets

2.2. Proposition. *$\mathcal{A}(t,p,N)$ satisfies the equation*

$$\mathcal{A}''(t,p,N) + \mathcal{R}(t)\mathcal{A}(t,p,N) = 0$$

with the initial conditions $\mathcal{A}(0,p,N) = \mathrm{Id}_q \oplus 0$ and $\mathcal{A}'(0,p,N) = (-L_N) \oplus \mathrm{Id}_{n-q-1}$, where Id_q means the identity map on T_pP, and Id_{n-q-1} and 0 are the identity and the zero map on $\mathcal{N}_pP \cap \{N\}^\perp$, respectively.

Let $\mathcal{S}$ be a subset of TM. From now on we shall denote by $\exp_{\mathcal{S}}$ the restriction to $\mathcal{S}$ of the exponential map on TM.

From Definition 2.1 and the relation between the transverse Jacobi vector fields and the exponential map (cf. [DC], for example), it is straightforward that

2.3. Proposition. *Let $\mathcal{N}P(t) = \{X \in \mathcal{N}P$ such that $|X| = t\}$. Then*

$$\operatorname{rank} \mathcal{A}(t,p,N) = \operatorname{rank}\{\exp_{\mathcal{N}P(t)}\}_{*tN}.$$

To end this paragraph we give an expression of the metric of M in terms of the operator $\mathcal{A}(t,p,N)$ and the metric of P, which generalizes those given in [Ch] for P a point and for P a real hypersurface (see also [K–V]). For this we first need a special type of coordinates: the spherical Fermi coordinates, which coincide with the spherical geodesic coordinates when P is a point and with Fermi coordinates when P is a hypersurface. Spherical Fermi coordinates are related with Fermi coordinates as spherical geodesic coordinates are related with geodesic coordinates.

2.4. Definition. Let (U,ϕ) and (V,ψ) be coordinate systems of P and S^{n-q-1} respectively. Let $\{f_k\}_{k=q+1,\ldots,n}$ be a local frame of $\mathcal{N}P$ on U. Let us define $N: U \times V \to \delta\mathcal{N}P$ by

$$N(p,\xi) = \sum_{k=q+1}^{n} \xi^k f_k(p),$$

where $\xi = (\xi^{q+1},\ldots,\xi^n) \in S^{n-q-1} \subset \mathbf{R}^{n-q}$. Let $W = \phi(U) \times \psi(V) \times I$ $(I \subset \mathbf{R}^+)$ and $x: W \to M$ defined by $x(u,v,t) = \exp_{\phi^{-1}(u)} tN(\phi^{-1}(u), \psi^{-1}(v))$. We can take

U, V, and I so that x is a diffeomorphism. Then the pair $(x(W), x^{-1})$ is called a system of spherical Fermi coordinates.

2.5. Proposition. *In a system of spherical Fermi coordinates the metric tensor has the expression given by*

$$\begin{aligned} ds^2 = dt^2 &+ \sum_{\alpha,\beta=1}^{q} \langle \mathfrak{A}(t)[\partial_\alpha \phi^{-1} + (\nabla_\alpha N)^\perp], \\ &\qquad \mathfrak{A}(t)[\partial_\beta \phi^{-1} + (\nabla_\beta N)^\perp]\rangle\, du^\alpha du^\beta \\ &+ \sum_{i,j=q+1}^{n-1} \langle \mathfrak{A}(t)\partial_i N, \mathfrak{A}(t)\partial_j N\rangle\, dv^i dv^j \\ &+ \sum_{\substack{\alpha=1,\ldots,q \\ i=q+1,\ldots,n-1}} \langle \mathfrak{A}(t)[\partial_\alpha \phi^{-1} + (\nabla_\alpha N)^\perp], \mathfrak{A}(t)\partial_i N\rangle\, du^\alpha dv^i \end{aligned} \tag{2.5.1}$$

where $\mathfrak{A}(t) = \mathfrak{A}(t, \phi^{-1}(u), N(\phi^{-1}(u), \psi^{-1}(v)))$, $\nabla_\alpha = \nabla_{\partial/\partial u^\alpha}$, $\nabla_i = \nabla_{\partial/\partial v^i}$, $\partial_\alpha = \partial/\partial u^\alpha$ *and* $\partial_i = \partial/\partial v^i$.

Proof. It is easy to see that $Y_\alpha(t) = \partial_\alpha x(u,v,t)$ and $Y_i(t) = \partial_i x(u,v,t)$ are transverse Jacobi fields along the geodesic $\gamma(t) = \exp_{\phi^{-1}(u)} tN(\phi^{-1}(u), \psi^{-1}(v))$ satisfying the initial conditions

$$Y_\alpha(0) = \partial_\alpha \phi^{-1}(u), \qquad Y'_\alpha(0) = \nabla_\alpha N(\phi^{-1}(u), \psi^{-1}(v)),$$

$$Y_i(0) = 0, \qquad Y'_i(0) = \partial_i N(\phi^{-1}(u), \psi^{-1}(v)).$$

On the other hand $\tilde{Y}_\alpha(t) = \tau_t \mathfrak{A}(t)[\partial_\alpha \phi^{-1} + (\nabla_\alpha N)^\perp]$ and $\tilde{Y}_i(t) = \tau_t \mathfrak{A}(t)\partial_i N$ are transverse Jacobi fields along $\gamma(t)$ such that

$$\tilde{Y}_\alpha(0) = \partial_\alpha \phi^{-1}(u), \qquad \tilde{Y}'_\alpha(0) + L_{\gamma'(0)}\partial_\alpha \phi^{-1}(u) = (\nabla_\alpha N)^\perp,$$

$$\tilde{Y}_i(0) = 0, \qquad \tilde{Y}'_i(0) = \partial_i N.$$

This implies $Y_i = \tilde{Y}_i$.

Moreover, since Y_α is a transverse Jacobi field along $\gamma(t)$, we have

$$Y'_\alpha(0) + L_{\gamma'(0)}\partial_\alpha \phi^{-1}(u) \in \mathfrak{N}_{\gamma(0)} P, \qquad Y'_\alpha(0) = \nabla_\alpha N;$$

and since $L_{\gamma'(0)}\partial_\alpha \phi^{-1}(u) \in T_{\gamma(0)} P$, we have $Y'_\alpha(0) + L_{\gamma'(0)}\partial_\alpha \phi^{-1}(u) = (\nabla_\alpha N)^\perp$. All this implies $Y_\alpha = \tilde{Y}_\alpha$.

On the other hand, by the Gauss lemma for tubes (see [Gr 1] or [Ch]), $\langle \partial_\alpha x, \partial_t x\rangle = \langle \partial_i x, \partial_t x\rangle = 0$. Then (2.5.1) follows.

§3. Estimates for the volume of complex submanifolds

From now on, M will be a connected compact Kaehler manifold of real dimension $2n$, and P will denote a closed complex submanifold of M of real dimension $2q$, $q > 0$. We shall denote by $\langle\ ,\ \rangle$ and J the metric and the complex structure of M, respectively.

Let $p \in P$, $N \in \mathcal{S}\mathfrak{N}_p P$ and $\gamma_N(t)$ as in 2.1. Let $f(N) = \inf\{t > 0 / \gamma_N(t)$ is a focal point of $P\}$. For every $t \in\,]0, f(N)[$, there is a neighbourhood U of $\gamma_N(t)$ and a neighbourhood V of p in P such that $P(t) = U \cap \{m \in M / d(m, V) = t\}$ is a real hypersurface of M. We shall denote by $S(t)$ the Weingarten map of $P(t)$ associated to a unit normal vector field N^t defined on $P(t)$ as an extension of $\gamma_N'(t)$. Let ω be the Riemannian volume element of M, dp that of P, and dN that of the unit sphere $S^{2n-2q-1}$. Let $\theta_N(p,t)$ be the real function defined on $\{(p,N,t) \in \mathcal{S}\mathfrak{N}P \times \mathbf{R} / 0 < t < f(N)\}$ by $\omega(\gamma_N(t)) = \theta_N(p,t)\, t^{2n-2q-1}\, dN \wedge dp \wedge dt$. In [Gr 1] it is proved that:

$$\theta_N'(p,t)/\theta_N(p,t) = -[\operatorname{tr} S(t) + (2n-2q-1)/t], \tag{3.0.1}$$

$$S'(t) = S(t)^2 + R(t) \tag{3.0.2}$$

(where $S'(t) = \nabla_{\gamma_N'(t)} S(t)$, $R(t)$ is defined in 2.1), and

$$\lim_{t \to 0} \theta_N(p,t) = 1. \tag{3.0.3}$$

Actually the function $\theta_N(p,t)$ is defined in [Gr 1] by the action of ω on certain Fermi fields. If we take the coordinates defined in 2.4 with $\{\partial_\alpha\}_{\alpha=1,\dots,2q}$ orthonormal at p and $\{\partial_i\}_{i=2q+1,\dots,2n-1}$ orthonormal at $\xi \in S^{2n-2q-1}$ such that $N(p,\xi) = N$, this yields

$$\theta_N(p,t) = \omega\left(\frac{\partial x}{\partial u^1}, \dots, \frac{\partial x}{\partial u^{2q}}, \frac{\partial x}{\partial t}, \frac{1}{t}\frac{\partial x}{\partial v^{2q+1}}, \dots, \frac{1}{t}\frac{\partial x}{\partial v^{2n-1}}\right).$$

Since $\partial_\alpha x$, $\partial_i x$ are Jacobi fields along $\gamma_N(t)$ and $\partial_t x = \gamma_N'(t)$, these vector fields can be extended along $\gamma_N(t)$ for every t. Then, for each $p \in P$, $N \in \mathcal{S}\mathfrak{N}_p P$, $\theta_N(p,t)$ is well defined and C^∞ on $\mathbf{R}$. In this paragraph we shall denote this extension also by $\theta_N(p,t)$. In §4 another extension will be considered.

We shall consider the following orthogonal direct decomposition of $T_{\gamma_N(t)}M = H_t \oplus \langle[\gamma_N'(t), J\gamma_N'(t)]\rangle \oplus V_t$, where $H_t = \tau_t T_p P$ and $\langle[\gamma_N'(t), J\gamma_N'(t)]\rangle$ is the vector space generated by $\gamma_N'(t)$ and $J\gamma_N'(t)$.

3.1. Lemma. *Suppose that, for every $p \in P$, $N \in \mathcal{S}\mathfrak{N}_p P$, $K_H(\gamma_N'(t)) \geq 4\lambda$, $K(\gamma_N'(t), H_t) \geq 2q\lambda$ and $K(\gamma_N'(t), V_t) \geq 2(n-q-1)\lambda$. Then*

$$\theta_N(p,t) \le \left(\frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}\,t}\right)^{2n-2q-1} \cos^{2q+1}(\sqrt{\lambda}\,t) \equiv \eta(t), \tag{3.1.1}$$

$$\theta_N(p,t)/\eta(t) \text{ is a decreasing function,} \tag{3.1.2}$$

and the equality holds in (3.1.1) *if and only if* $L_N = 0$ *for every* $N \in \mathcal{S}_p\mathfrak{N}P$ *and* $\mathcal{R}(t)$ *and* $\mathcal{A}(t,p,N)$ *have the matrix expression*

$$\mathbf{R}(t) = \begin{bmatrix} \lambda & & & 0 \\ & \ddots_{2n-2} & & \\ & & \lambda & \\ 0 & & & 4\lambda \end{bmatrix}, \tag{3.1.3}$$

$$\mathcal{A}(t,P,N) = \begin{bmatrix} \cos(\sqrt{\lambda}\,t) & & & & & & 0 \\ & \ddots_{2q} & & & & & \\ & & \cos(\sqrt{\lambda}\,t) & & & & \\ & & & \frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}} & & & \\ & & & & \ddots_{2n-2q-2} & & \\ & & & & & \frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}} & \\ 0 & & & & & & \frac{\sin(2\sqrt{\lambda}\,t)}{2\sqrt{\lambda}} \end{bmatrix},$$

with respect to an orthonormal basis $\{e_1, e_2 = Je_1, \ldots, e_{2n-3}, e_{2n-2} = Je_{2n-3}, e_{2n-1} = JN\}$ *of* $\{\gamma_N'(0)\}^{\perp}$ *such that* $\{e_i\}_{i=1,\ldots,2q}$ *is a basis of* T_pP.

If $M = \mathbf{C}P^n(\lambda)$ *and* $P = \mathbf{C}P^q(\lambda)$, *then* $\theta_N(p,t) = \eta(t)$.

PROOF. Inequality (3.1.1) is essentially proved in [Gr 2]. We give here a slightly different proof in order to make it more obvious to the reader that equality implies (3.1.3).

Let $\{E_i(t)\}_{1\le i\le 2n-1}$ be a basis of $\{\gamma_N'(t)\}^{\perp}$ obtained by parallel transport along $\gamma_N(t)$ of an orthonormal basis $\{e_i\}_{1\le i\le 2n-1}$ of T_pP which diagonalizes the Weingarten map L_N of P associated to N. Let us consider the functions $f_i(t) =$

$\langle S(t)E_i(t), E_i(t)\rangle$. Taking the derivatives of f_i, using (3.0.1) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} f_i' &= \langle S'E_i, E_i\rangle = \langle S^2E_i + R(t)E_i, E_i\rangle \\ &= \|SE_i\|^2 + \langle R(t)E_i, E_i\rangle \geq \langle SE_i, E_i\rangle^2 + \langle R(t)E_i, E_i\rangle \\ &= f_i^2 + \langle R(t)E_i, E_i\rangle. \end{aligned} \tag{3.1.4}$$

Then, using the hypothesis $K_H(\gamma_N'(t)) \geq 4\lambda$, $K(\gamma_N'(t), H_t) \geq 2q\lambda$ and $K(\gamma_N'(t), V_t) \geq 2(n-q-1)\lambda$, we get

$$\begin{aligned} &\left(\frac{1}{2q}\sum_{i=1}^{2q} f_i\right)' \geq \left(\frac{1}{2q}\sum_{i=1}^{2q} f_i^2\right) + \lambda \geq \left(\frac{1}{2q}\sum_{i=1}^{2q} f_i\right)^2 + \lambda, \\ &\left(\frac{1}{2n-2q-2}\sum_{i=2q+1}^{2n-2} f_i\right)' \geq \left(\frac{1}{2n-2q-2}\sum_{i=2q+1}^{2n-2} f_i\right)^2 + \lambda, \end{aligned} \tag{3.1.5}$$

and

$$f_{2n-1}' \geq f_{2n-1}^2 + 4\lambda.$$

For $i = 1, \ldots, 2q$ we have that, if k_i is the principal curvature of L_N in the direction of e_i, $f_i(0) = k_i$. Moreover, $k_{2j} = -k_{2j-1}$ for $1 \leq j \leq q$, because P is a complex submanifold. Then

$$\left(\frac{1}{2q}\sum_{i=1}^{2q} f_i\right)(0) = 0$$

and, from [Gr 1, Lemma 5.1] it follows that

$$\frac{1}{2q}\sum_{i=1}^{2q} f_i(t) \geq \sqrt{\lambda}\tan(\sqrt{\lambda}t). \tag{3.1.6}$$

Then

$$\sum_{i=1}^{2q} f_i(t) \geq -\frac{d}{dt}\ln(\cos^{2q}(\sqrt{\lambda}t)). \tag{3.1.7}$$

For $i = 2q+1, \ldots, 2n-1$ we have $f_i(0) = -\infty$. Then, from (3.1.5) and [Gr 1, Lemma 5.2],

$$\sum_{i=2q+1}^{2n-2} f_i(t) \geq -\frac{2(n-q-1)\sqrt{\lambda}}{\tan(\sqrt{\lambda}\,t)} \geq -\frac{d}{dt}\ln\left[\sin^{2n-2q-2}(\sqrt{\lambda}\,t)\right]$$

and

$$f_{2n-1}(t) \geq -\frac{2\sqrt{\lambda}}{\tan(2\sqrt{\lambda}\,t)} = -\frac{d}{dt}\ln[\sin(2\sqrt{\lambda}\,t)].$$

Then, by (3.0.1),

$$\frac{d}{dt}\ln\theta_N(p,t) = -[\operatorname{tr} S(t) + (2n-2q-1)/t] = -\sum_{i=1}^{2n-1} f_i(t) - \frac{(2n-2q-1)}{t}$$
$$\leq \frac{d}{dt}\ln\left[\cos^{2q+1}(\sqrt{\lambda}\,t)\left(\frac{\sin(\sqrt{\lambda}\,t)}{t}\right)^{2n-2q-1}\right],$$

from which (3.1.2) and (3.1.1) follow, having account of (3.0.3).

If we have the equality in (3.1.1), then all the above inequalities must be equalities, which implies that $k_i = 0$ for $i = 1,\ldots,2q$, and $\{E_i(t)\}_{1\leq i\leq 2n-1}$ diagonalizes $S(t)$ with eigenvalues

$$f_i(t) = \begin{cases} \sqrt{\lambda}\tan(\sqrt{\lambda}\,t) & \text{for } i = 1,\ldots,2q, \\ -\sqrt{\lambda}\cot(\sqrt{\lambda}\,t) & \text{for } i = 2q+1,\ldots,2n-2, \\ -2\sqrt{\lambda}\cot(2\sqrt{\lambda}\,t) & \text{for } i = 2n-1. \end{cases}$$

Then the expression (3.1.3) for $\mathfrak{R}(t)$ follows from (3.0.2), and the expression for $\mathfrak{A}(t,p,N)$ follows from Proposition 2.2.

If $M = \mathbf{C}P^n(\lambda)$ and $P = \mathbf{C}P^q(\lambda)$, then all the above inequalities are equalities, and $\theta_N(p,t) = \eta(t)$.

3.2. Corollary. *For each $N \in \mathfrak{s}\mathfrak{N}P$, let $c(N) = \sup\{t > 0/d(P,\gamma_N(t)) = t\}$ and $c(P) = \inf\{c(N)/N \in \mathfrak{s}\mathfrak{N}P\}$. If $K_H \geq 4\lambda$ and $K_A \geq \lambda$, then $c(P) \leq c(N) \leq \pi/2\sqrt{\lambda}$. If $M = \mathbf{C}P^n(\lambda)$ and $P = \mathbf{C}P^q(\lambda)$, then $c(P) = c(N) = \pi/2\sqrt{\lambda}$ $\forall N \in \mathfrak{s}\mathfrak{N}\mathbf{C}P^q(\lambda)$.*

Proof. Given a function $q: X \times \mathbf{R}^+ \to \mathbf{R}$, where X is a given space, let us denote by $z(q)$ the function which, to every $x \in X$, associates the first zero of the function $t \to q(x,t)$. Then, it is well known that $c(N) \leq f(N) = z(\theta_N(p,t))$. Then, by (3.1.1) we have that $c(P) \leq c(\ ) \leq z(\eta(t)) = \pi/2\sqrt{\lambda}$.

3.3. Theorem. *Let M and P be as in* 3.1. *Then*

(3.3.1) $$\frac{\operatorname{vol}(P)}{\operatorname{vol}(M)} \geq \frac{\operatorname{vol}(\mathbf{C}P^q(\lambda))}{\operatorname{vol}(\mathbf{C}P^n(\lambda))}.$$

PROOF. Let us denote by dN the volume element of the euclidean sphere of radius one $S^{2n-2q-1} \subset \mathcal{S}\mathfrak{N}_p P$. If $\mathrm{cut}(P) = \{\gamma_N(c(N)),\ N \in \mathcal{S}\mathfrak{N}P\}$, we have that $M = \{\gamma_N(t),\ N \in \mathcal{S}\mathfrak{N}P,\ 0 \le t < c(N)\} \cup \mathrm{cut}(P)$. Then, from (3.1.1) and 3.2, we have

$$\begin{aligned}
\mathrm{vol}(M) &= \int_P \int_0^{c(N)} \int_{S^{2n-2q-1}} t^{2n-2q-1}\theta_N(p,t)\, dN\, dp\, dt \\
&\le \int_0^{\pi/2\sqrt{\lambda}} \int_P \int_{S^{2n-2q-1}} t^{2n-2q-1}\eta(t)\, dN\, dp\, dt \\
&= \frac{\mathrm{vol}(P)}{\mathrm{vol}(\mathbf{C}P^q(\lambda))} \int_0^{\pi/2\sqrt{\lambda}} \int_{\mathbf{C}P^q(\lambda)} \int_{S^{2n-2q-1}} t^{2n-2q-1}\eta(t)\, dN\, dp\, dt \\
&= \frac{\mathrm{vol}(P)\mathrm{vol}(\mathbf{C}P^n(\lambda))}{\mathrm{vol}(\mathbf{C}P^q(\lambda))}
\end{aligned}$$

3.4. THEOREM. *Let M and P be as in* 3.3, *with $q = n-1$. Then the equality in* (3.3.1) *holds if and only if there is a holomorphic isometry $i : M \to \mathbf{C}P^n(\lambda)$ such that $i(P) = \mathbf{C}P^{n-1}(\lambda)$.*

PROOF. If we have the equality in (3.3.1), then (3.1.1) must have an equality, which implies that $\mathfrak{A}(t,p,N)$ has the form given in (3.1.3). Then $\mathfrak{A}(t,p,N)$ has rank $2n-1$ for $0 < t < \pi/(2\sqrt{\lambda})$ and rank 0 for $t = \pi/(2\sqrt{\lambda})$. Then, by 2.3,

$$\mathrm{rank}\{\exp_{\mathfrak{N}P(\pi/(2\sqrt{\lambda}))}\}_{*(\pi/(2\sqrt{\lambda}))N} = 0 \qquad \text{for every } N \in \mathcal{S}\mathfrak{N}P.$$

Then, since $\mathfrak{N}P(\pi/(2\sqrt{\lambda}))$ is connected, there is a point $m \in M$ such that

$$\exp_{\mathfrak{N}P}(\{\pi/(2\sqrt{\lambda})N / N \in \mathcal{S}\mathfrak{N}P\}) = \{m\}. \tag{3.4.1}$$

Since equality in (3.3.1) implies also that $c(N) = \pi/(2\sqrt{\lambda})$ for every $N \in \mathcal{S}\mathfrak{N}P$, (3.4.1) means that m can be joined to every point $p \in P$ by a minimizing geodesic orthogonal to P.

Let $\Phi : \mathcal{S}\mathfrak{N}P \to S^{2n-1} \subset T_m M$ be the continuous map given by

$$\Phi(N) = \gamma_N'(\pi/(2\sqrt{\lambda})) = \left(\frac{d}{dt}\exp_{\mathfrak{N}P} tN\right)\left(\frac{\pi}{2\sqrt{\lambda}}\right).$$

Since $\mathcal{S}\mathfrak{N}P$ is compact we have that $\Phi(\mathcal{S}\mathfrak{N}P)$ is closed in S^{2n-1}. But if $\xi \in \Phi(\mathcal{S}\mathfrak{N}P)$, there is an $N \in \mathcal{S}\mathfrak{N}P$ such that $\gamma_N'(\pi/(2\sqrt{\lambda})) = \xi$, and we have that

$$\exp_m(-t\xi) = \gamma_N\left(\frac{\pi}{2\sqrt{\lambda}} - t\right).$$

Then

$$V \equiv \exp_{\mathfrak{N}P}\left(\left\{tN/0 < t < \frac{\pi}{2\sqrt{\lambda}}, N \in \mathfrak{s}\mathfrak{N}P\right\}\right)$$
$$= \exp_m\left(\left\{s\xi/0 < s < \frac{\pi}{2\sqrt{\lambda}}, \xi \in \Phi(\mathfrak{s}\mathfrak{N}P)\right\}\right).$$

Let

$$\mathfrak{J} :]0, \pi/(2, \sqrt{\lambda})[\times \Phi(\mathfrak{s}\mathfrak{N}P) \to \left\{s\xi/0 < s < \frac{\pi}{2\sqrt{\lambda}}, \xi \in \Phi(\mathfrak{s}\mathfrak{N}P)\right\}$$

be the diffeomorphism given by $\mathfrak{J}(s, \xi) = s\xi$. Then $]0, \pi/(2\sqrt{\lambda})[\times \Phi(\mathfrak{s}\mathfrak{N}P) = \mathfrak{J}^{-1} \cdot \exp_m^{-1}(V)$, which is open. Then $\Phi(\mathfrak{s}\mathfrak{N}P)$ is open in S^{2n-1}. Then, since S^{2n-1} is connected, $\Phi(\mathfrak{s}\mathfrak{N}P) = S^{2n-1}$. Then every geodesic $\exp_m(-t\xi)$ starting from m ($\xi \in S^{2n-1}$) will meet a point of P for $t = \pi/(2\sqrt{\lambda})$ and there is no cut point of m before P, i.e. $\exp_m$ is a diffeomorphism from the open ball of center 0 and radius $\pi/(2\sqrt{\lambda})$ in T_mM into $M - P$, and P is the set of cut points of m (since $K_H(\gamma_N'(t)) \geq 4\lambda$ and $\rho_A(\gamma_N'(t)) \geq 2(n-1)\lambda$ imply $c(\xi) \leq \pi/(2\sqrt{\lambda})$ for every ξ).

Moreover, from the fact that $\Phi(\mathfrak{s}\mathfrak{N}P) = S^{2n-1}$ we have that for every $\xi \in S^{2n-1}$ there is an $N \in \mathfrak{s}\mathfrak{N}P$ such that

$$\gamma_N\left(\frac{\pi}{2\sqrt{\lambda}} - t\right) = \exp_m(-t\xi) \equiv \gamma_\xi(t).$$

Then, if $R_m(t)$ is the operator along $\gamma_{-\xi}(t)$ defined in 2.1, we have that

$$R_m(t) = R\left(\frac{\pi}{2\sqrt{\lambda}} - t\right).$$

Then, solving the differential equation given in Proposition 2.2, we get (here $q = 0$)

$$\mathfrak{A}(t, m, \xi) = \begin{bmatrix} \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} & & & 0 \\ & \ddots & & \\ & & \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} & \\ 0 & & & \frac{\sin(2\sqrt{\lambda}\, t)}{2\sqrt{\lambda}} \end{bmatrix}. \tag{3.4.2}$$

Now, let $m' \in \mathbf{C}P^n(\lambda)$, and $j : T_mM \to T_{m'}\mathbf{C}P^n(\lambda)$ a holomorphic isometry, and let us define $i' : M - P \to \mathbf{C}P^n(\lambda) - \mathbf{C}P^{n-1}(\lambda)$ by

$$i'(\exp_m s\xi) = \exp_{m'} sj(\xi), \qquad \text{for } 0 \leq s \leq \pi/2\sqrt{\lambda}, \quad \xi \in S^{2n-1}.$$

We have seen that P is the cut locus of m, and it is well known that the cut locus of m' is a $\mathbf{C}P^{n-1}(\lambda)$. On the other hand, $R_{m'}(t)$ has also the expression (3.1.3) for every geodesic $\gamma(t)$ of $\mathbf{C}P^n(\lambda)$ starting at m'. Then $\mathfrak{A}(t,m',j(\xi))$ has also the matrix expression (3.4.2) in $\mathbf{C}P^n(\lambda)$. Then, by 2.5, the map i' is a holomorphic isometry. Then $M-P$ has constant holomorphic sectional curvature. Moreover, $M-P$ is dense in M, whence by continuity M has constant holomorphic sectional curvature, and, M being compact, there is a holomorphic isometry $i:M\to\mathbf{C}P^n(\lambda)$ (cf. [K-N]). If we take $m'=i(m)$ and $j=i_*$, then $i_{|M-P}=i'$, which implies that $i(P)=\mathbf{C}P^{n-1}(\lambda)$.

3.5. A crucial point in the proof of Theorem 3.4 is the fact that $\operatorname{cut}(P)$ is a point. Then this proof works only for complex hypersurfaces. For $q<n-1$, $\operatorname{cut}(P)$ is an immersed submanifold of M, and if we want to prove the isometry between M and $\mathbf{C}P^n(\lambda)$ by using 2.5, we must prove that $\operatorname{cut}(P)$ is isometric to $\operatorname{cut}(\mathbf{C}P^q(\lambda))=\mathbf{C}P^{n-q-1}(\lambda)$, which seems as difficult as proving the isometry between P and $\mathbf{C}P^q(\lambda)$. The situation is easier if we fix M or P. For example, if $M=\mathbf{C}P^n(\lambda)$, Theorem 3.3 implies that $\operatorname{vol}(P)\geq\operatorname{vol}(\mathbf{C}P^q(\lambda))$ and equality implies, by 3.1, that P is totally geodesic in $\mathbf{C}P^n(\lambda)$, then (cf. [Ki]) P is $\mathbf{C}P^q(\lambda)$. Another possible situation consists in fixing $P=\mathbf{C}P^q(\lambda)$ and determining M by the equality in (3.3.1), but this is an obvious consequence of [Na, th. 2]). Let P_r be the tube of radius r about P in M. If we take the function $\operatorname{vol}(P)/\operatorname{vol}(P_r)$ instead of $\operatorname{vol}(P)/\operatorname{vol}(M)$ and fix $P=\mathbf{C}P^q(\lambda)$, the determination of P_r by the corresponding equality on the above function is not so obvious. We shall do that in the next section.

§4. A theorem of Gromov–Nayatani type for tubes about complex submanifolds

In this paragraph a tube P_r (respectively the boundary ∂P_r of a tube) of radius r about P in M will mean the set of points $m\in M$ such that $d(m,P)\leq r$ (respectively $d(m,P)=r$). With this definition, if $r>c\equiv\max\{c(N),\,N\in\mathcal{S}\mathfrak{N}P\}$, then $P_c=P_r$ and $\partial P_r=\varnothing$.

In order that the usual formulas computing the volume of P_r remain valid, we shall extend the function $\theta_N(p,t)$ (respectively $\eta(t)$), defined in §3, by $\theta_N(p,t)=0$ when $t>c(N)$ (respectively by $\eta(t)=0$ when $t>\pi/2\sqrt{\lambda}$). Then we have

4.1. Theorem. *Let M and P be as in* 3.1. *Then the functions*

$$f(r) = \frac{\mathrm{vol}(P_r)}{\mathrm{vol}[\mathbf{C}P^q(\lambda)_r]}, \quad g(r) = \frac{\mathrm{vol}(M - P_r)}{\mathrm{vol}[\mathbf{C}P^n(\lambda) - \mathbf{C}P^q(\lambda)_r]},$$

$$h(r) = \frac{\mathrm{vol}(\partial P_r)}{\mathrm{vol}[\partial(\mathbf{C}P^q(\lambda)_r]}$$

defined on $\mathbf{R}^+$ *are decreasing and*

$$\frac{\mathrm{vol}(P)}{\mathrm{vol}[\mathbf{C}P^q(\lambda)]} = f(0) \geq f(r) \geq h(r) \geq g(r) \qquad \text{for every } r \in \mathbf{R}^+. \tag{4.1.1}$$

Moreover $f(r) = h(r)$ (*respectively* $g(r) = h(r)$) *if and only if* r *is a critical point of* f (*respectively* g).

PROOF. The extensions taken of $\theta_N(p,t)$ and $\eta(t)$ allow the statement of the theorem to be valid also for $r \geq \mathrm{cut}(P)$.

Now we are proving that the above functions are not increasing:

(a) *f is not increasing*:

Observe that

$$\mathrm{vol}(P_r) = \int_P \int_{S^{2n-2q-1}} \int_0^r t^{2n-2q-1} \theta_N(p,t)\, dt\, dN\, dp,$$

$$\begin{aligned} \mathrm{vol}[\mathbf{C}P^q(\lambda)_r] &= \int_{\mathbf{C}P^q(\lambda)} \int_{S^{2n-2q-1}} \int_0^r t^{2n-2q-1} \eta(t)\, dt\, dN\, dp \\ &= \mathrm{vol}[\mathbf{C}P^q(\lambda)] \mathrm{vol}(S^{2n-2q-1}) \int_0^r t^{2n-2q-1} \eta(t)\, dt. \end{aligned}$$

Let

$$g_{N,p}(r) = \frac{\displaystyle\int_0^r t^{2n-2q-1} \theta_N(p,t)\, dt}{\displaystyle\int_0^r t^{2n-2q-1} \eta(t)\, dt}.$$

For $0 < r < \pi/2\sqrt{\lambda}$, the derivative of this function satisfies

$$g'_{N,p}(r) = \frac{r^{2n-2q-1}\eta(r)\left[\displaystyle\int_0^r t^{2n-2q-1}\left\{\frac{\theta_N(p,r)}{\eta(r)}\,\eta(t) - \theta_N(p,t)\right\} dt\right]}{\left(\displaystyle\int_0^r t^{2n-2q-1}\eta(t)\, dt\right)^2} \leq 0$$

since, by (3.1.2),

$$\frac{\theta_N(p,r)}{\eta(r)} \leq \frac{\theta_N(p,t)}{\eta(t)} \qquad \text{for } 0 \leq t \leq r.$$

For $r \geq \pi/2\sqrt{\lambda}$: $g'_{N,p}(r) = 0$.

Then $g_{N,p}(r)$ is not increasing for every real r. From this it follows that, if $r < r'$, then

$$\frac{\mathrm{vol}(P_r)}{\mathrm{vol}[\mathbf{C}P^q(\lambda)_r]} = \frac{\displaystyle\int_P \int_{S^{2n-2q-1}} g_{N,p}(r)\, dN\, dp}{\mathrm{vol}[\mathbf{C}P^q(\lambda)]\mathrm{vol}(S^{2n-2q-1})}$$

$$\geq \frac{\displaystyle\int_P \int_{S^{2n-2q-1}} g_{N,p}(r')\, dN\, dp}{\mathrm{vol}[\mathbf{C}P^q(\lambda)]\mathrm{vol}(S^{2n-2q-1})} = \frac{\mathrm{vol}(P_{r'})}{\mathrm{vol}[\mathbf{C}P^q(\lambda)_{r'}]}.$$

(b) For the function g the same proof works with

$$\mathrm{vol}(M - P_r) = \int_P \int_{S^{2n-2q-1}} \int_r^{\pi/2\sqrt{\lambda}} t^{2n-2q-1}\theta_N(p,t)\, dt\, dN\, dp,$$

$$\mathrm{vol}[\mathbf{C}P^n(\lambda) - \mathbf{C}P^q(\lambda)_r] = \int_{\mathbf{C}P^q(\lambda)} \int_{S^{2n-2q-1}} \int_r^{\pi/2\sqrt{\lambda}} t^{2n-2q-1}\eta(t)\, dt\, dN\, dp,$$

$$g_{N,p}(r) = \frac{\displaystyle\int_r^{\pi/2\sqrt{\lambda}} t^{2n-2q-1}\theta_N(p,t)\, dt}{\displaystyle\int_r^{\pi/2\sqrt{\lambda}} t^{2n-2q-1}\eta(t)\, dt}.$$

(c) For the function h,

$$\mathrm{vol}(\partial P_r) = \int_P \int_{S^{2n-2q-1}} r^{2n-2q-1}\theta_N(p,r)\, dN\, dp,$$

$$\mathrm{vol}[\partial \mathbf{C}P^q(\lambda)_r] = \int_{\mathbf{C}P^q(\lambda)} \int_{S^{2n-2q-1}} r^{2n-2q-1}\eta(r)\, dN\, dp,$$

$$h(r) = \frac{\displaystyle\int_P \int_{S^{2n-2q-1}} \theta_N(p,r)\, dN\, dp}{\eta(r)\mathrm{vol}[\mathbf{C}P^q(\lambda)]\mathrm{vol}(S^{2n-2q-1})},$$

$$h'(r) = \frac{\mathrm{vol}(\mathbf{C}P^q(\lambda))\mathrm{vol}(S^{2n-2q-1}) \displaystyle\int_P \int_{S^{2n-2q-1}} [\theta'_N(p,r)\eta(r) - \theta_N(p,r)\eta'(r)]\, dN\, dp}{(\eta(r)\mathrm{vol}(\mathbf{C}P^q(\lambda))\mathrm{vol}(S^{2n-2q-1}))^2}.$$

Then $h'(r) \leq 0$ because $0 \geq (\theta/\eta)' = (\theta'\eta - \theta\eta')/\eta^2$ for $0 \leq r < \pi/2\sqrt{\lambda}$ and $\theta'\eta - \theta\eta' = 0$ for $r \geq \pi/2\sqrt{\lambda}$.

Now we are proving the inequalities between the functions f, g, h. It is well known ([Gr 2]) that

$$\frac{d}{dr}\operatorname{vol}(P_r) = \operatorname{vol}(\partial P_r), \qquad \frac{d}{dr}\operatorname{vol}(\mathbf{C}P^q(\lambda)_r) = \operatorname{vol}(\partial \mathbf{C}P^q(\lambda)_r).$$

Since f is not increasing, we have

$$0 \geq \frac{d}{dr}\left[\frac{\operatorname{vol}(P_r)}{\operatorname{vol}(\mathbf{C}P^q(\lambda)_r)}\right] = \frac{\operatorname{vol}(\partial P_r)\operatorname{vol}(\mathbf{C}P^q(\lambda)_r) - \operatorname{vol}(P_r)\operatorname{vol}(\partial\mathbf{C}P^q(\lambda)_r)}{(\operatorname{vol}(\mathbf{C}P^q(\lambda)_r))^2},$$

and this implies $h \leq f$. The implication of the equality is obvious.

Analogously, by taking the derivative of g we get $g \leq h$.

4.2. COROLLARY. *Under the conditions of Theorem* 4.1 *we have*

$$\operatorname{vol}(M - P_r) \leq \operatorname{vol}(\mathbf{C}P^n(\lambda) - \mathbf{C}P^q(\lambda)_r). \tag{4.2.1}$$

The equality holds if and only if M is isometric to $\mathbf{C}P^n(\lambda)$ and P is isometric to $\mathbf{C}P^q(\lambda)$.

PROOF. For $0 \leq r$, $g(0) \geq g(r)$, and from [Na, Th. 2] $\operatorname{vol}(M) \leq \operatorname{vol}(\mathbf{C}P^n(\lambda))$. Then

$$1 \geq \frac{\operatorname{vol}(M)}{\operatorname{vol}(\mathbf{C}P^n(\lambda))} \geq \frac{\operatorname{vol}(M - P_r)}{\operatorname{vol}(\mathbf{C}P^n(\lambda) - \mathbf{C}P^q(\lambda)_r)}.$$

The equality in (4.2.1) implies $\operatorname{vol}(M) = \operatorname{vol}(\mathbf{C}P^n(\lambda))$, then ([Na, Th. 2]) M is isometric to $\mathbf{C}P^n(\lambda)$. Moreover $\theta_N(p,t) = \eta(t)$, $0 \leq t \leq \pi/2\sqrt{\lambda}$. Then P as the principal curvatures zero and, by [Ki], P is isometric to $\mathbf{C}P^q(\lambda)$.

4.3. COROLLARY. *Let $p_0 \in P$, $p_0' \in \mathbf{C}P^q(\lambda)$. Let us denote by* $\operatorname{cut}(p_0)$ *the cut locus of p_0. If* $\operatorname{vol}(P) = \operatorname{vol}(\mathbf{C}P^q(\lambda))$, $q > 0$, *then the equality*

$$\operatorname{vol}(P_r) = \operatorname{vol}(\mathbf{C}P^q(\lambda)_r) \tag{4.3.1}$$

implies that there is a holomorphic isometry between the vector bundles $\mathfrak{N}(P - \operatorname{cut}(p_0))$ *and* $\mathfrak{N}(\mathbf{C}P^q(\lambda) - \operatorname{cut}(p_0'))$. *If this isometry preserves the normal connections induced on these normal vector bundles by the Levi-Civita connection of M and $\mathbf{C}P^n(\lambda)$ respectively, then* $(P - \operatorname{cut}(p_0))_r$ *is holomorphically isometric to* $(\mathbf{C}P^q(\lambda) - \operatorname{cut}(p_0'))_r$.

PROOF. The equality (4.3.1) implies the equality in the first inequality in (4.1.1) and, then, since $f(t)$ is not increasing, that $f(t)$ is constant for $0 \leq t \leq r$. But, looking at the proof of 5.1, we see that this is equivalent to the statement that $\theta_N(p,t)/\eta(t)$ be constant for $0 \leq t \leq r$. As $\lim_{t\to 0}(\theta_N(p,t)/\eta(t)) = 1$, we get $\theta_N(p,t) = \eta(t)$ for $0 \leq t \leq r$. By Lemma 3.1, $\mathcal{R}(t)$ has the expression given in (3.1.3), and the Weingarten map L_N is zero for every $N \in \mathfrak{s}\,\mathfrak{N}_pP$, $p \in P$. Then P is a totally geodesic submanifold of M and both have the same bounds for their sectional curvatures. Then, it follows from [Na, th. 2] that $\mathrm{vol}(P) \leq \mathrm{vol}(\mathbf{C}P^q(\lambda))$ and the equality of the volumes of P and $\mathbf{C}P^q(\lambda)$ implies the existence of a holomorphic isometry $i: P \to \mathbf{C}P^q(\lambda)$. Since $\mathbf{C}P^q(\lambda)$ is a homogeneous space, we can assume that $i(p_0) = p_0'$. Let us define

$$\chi : \mathfrak{N}(P - \mathrm{cut}(p_0)) \to \mathfrak{N}(\mathbf{C}P^q(\lambda) - \mathrm{cut}(p_0'))$$

by

$$\chi(N) = \tau_t' i_{*p_0}(\tau_t^{-1}N), \tag{4.3.2}$$

where τ_t' (respectively τ_t) is the parallel transport in $\mathbf{C}P^n(\lambda)$ (respectively M) along the minimizing geodesic joining p_0' and $i(p)$ (respectively p_0 and p). As i is a holomorphic isometry and P and $\mathbf{C}P^q(\lambda)$ are totally geodesic submanifolds, it follows that χ is a holomorphic isometry of vector bundles.

By Proposition 2.2 the operator $\mathcal{A}(t,\gamma_N(0),\gamma_N'(0))$ satisfies the differential equation $\mathcal{A}'' \to \mathcal{R}\mathcal{A} = 0$ with the initial conditions $\mathcal{A}(0) = \mathrm{Id}_{2q} \oplus 0_{2n-2q-1}$, $\mathcal{A}'(0) = 0_{2q} \oplus \mathrm{Id}_{2n-2q-1}$ and $\mathcal{R}(t) = \mathrm{Id}_{2n-2} \oplus (4\,\mathrm{Id}_1)$ on the space $\mathcal{U} = \{N\}^\perp = \{JN\}^{\perp\mathcal{U}} \oplus \langle\{JN\}\rangle$, where $\perp$ and $\perp\mathcal{U}$ denote the orthogonal complement in $T_{\gamma_N(0)}M$ and in $\mathcal{U}$ respectively, and $\langle\{JN\}\rangle$ is the subspace generated by JN. Then

$$\mathcal{A}(t,p,N) = \mathcal{A}(t,i(p),\chi(N)). \tag{4.3.3}$$

Let $\{e_k\}_{k=2q+1,\dots,2n}$ be an orthonormal J-basis of $\mathfrak{N}_{p_*}P$. By parallel transport along the geodesics starting from p_0, we get an orthonormal J-frame $\{f_k\}_{k=2q+1,\dots,2n}$ of $\mathfrak{N}(P - \mathrm{cut}(p_0))$ and another one $\{f_k' = \chi(f_k)\}_{k=2q+1,\dots,2n}$ of $\mathfrak{N}(\mathbf{C}P^q(\lambda) - \mathrm{cut}(p_0'))$. Let us take these J-frames to define systems of spherical Fermi coordinates of M and $\mathbf{C}P^n(\lambda)$ respectively. It follows from the existence of the isometry i, (4.3.3), Proposition 2.5, and the hypothesis on the conservation of the normal connections by χ that the expression of the metric tensor in the coordinates just defined before is the same in M as in $\mathbf{C}P^n(\lambda)$.

Let us consider the map $\Phi : (P - \mathrm{cut}(p_0))_r \to (\mathbf{C}P^q(\lambda) - \mathrm{cut}(p_0'))_r$ defined by

$$\Phi(\exp_{\mathfrak{N}P} tN) = \exp_{\mathfrak{N}\mathbf{C}P^q(\lambda)} t\chi(N).$$

When we read it in the above spherical Fermi coordinates it is represented by the identity. Then the above remarks show that if χ preserves the normal connections in $\mathfrak{N}P$ and $\mathfrak{N}CP^q(\lambda)$, then Φ is a holomorphic isometry.

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